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INVESTIGATION OF SELFSIMILAR SOLUTIONS DESCRIBING FLOWS IN MIXING LAYERS*

V.N. DIYESPEROV

A complete investigation is made of the selfsimilar solutions of the boundary layer equation for the stream function with zero pressure gradient. They are a good description of the flow pattern in mixing layers since far from the separation point the latter is formed mainly under the effect of the boundary conditions and depends slightly on the initial conditions. The selfsimilar function $\Phi(\zeta; m)$ (ζ is the selfsimilar variable, and m the selfsimilarity parameter) satisfies a well-known third-order non-linear differential equation. It is successfully reduced to a first-order equation /1/, which enables us to investigate the behaviour of all the integral curves of $\Phi(\zeta; m)$ and, in particular, the examination of the question of the existence and uniqueness of the solutions of the two- and three-point problems that occur in the theory of displacement layers. For $m = 1$ these are classical problems /2—4/ and the Blasius boundary layer problem and for $m = 2$ the Goldstein problem for the wake /5/. The mixing layer encountered in the theory separations /6—11/ refers to the case $m \in (1, 2]$. The case $m = \infty$ occurs in the theory of non-stationary separation /12/.

From the viewpoint of the behaviour of the integral curves, the cases $m > 1$ and $0 < m \leqslant 1$ differ substantially. For $0 < m \leqslant 1$ their pattern is reformed in such a manner that solutions describing the flows in mixing layers with reverse velocities do not occur. Examples of the latter are given in /13, 14/.

To a first approximation the flow in a mixing layer is described by the equation for the stream function

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\,\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \lambda\frac{\partial^3\psi}{\partial y^3} \tag{1}$$

For an incompressible fluid $\lambda = 1$. For a gas $\lambda = \theta/R^2\ (0)$ in the theory of local

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separation, where $R(0)$ is the value of the density at the point of separation and θ is the Chapman constant in the linear dependence between the viscosity coefficient and the temperature. The (x, y) system of coordinates is orthogonal. Its selection depends on the problem under consideration.

The solutions of (1) are represented in the class of selfsimilar solutions in the form

$$\psi = \omega^{-1/2} x^{\omega\lambda} \Phi(\zeta), \quad \zeta = \omega^{1/2} y / x^{1/(m+1)}, \quad m > 0, \quad \lambda\omega = m/(m+1)$$

A non-linear third-order differential equation is obtained to determine $\Phi(\zeta)$:

$$\frac{m-1}{m}\Phi'^2 - \Phi\Phi'' = \Phi''' \tag{2}$$

The flow in a mixing layer originating during the interaction of two streams, one of which (the upper) moves while the other is at rest, is described by (2) and the three boundary conditions

$$\Phi = b\zeta^m + \ldots, \zeta \to +\infty, \; b > 0, \; m > 0 \tag{3}$$

$$\Phi = 0, \quad \zeta = 0 \tag{4}$$

$$d\Phi/d\zeta \to 0, \; \zeta \to -\infty \tag{5}$$

Problem (2)-(5) for $m = 2$ arises in both the theory of local separation from a smooth surface of an incompressible fluid $(\lambda = 1)$ /7/, and in a supersonic stream /6, 9/. Separation from the angular point of an incompressible fluid $(\lambda = 1)$ corresponds to $m = 5/3$ /8/, and a gas at sonic velocity $m = 7/6$ /11/. For $m = 1$ the classical Chapman problem is obtained /4/.

If

$$d^2\Phi/d\zeta^2 = 0, \quad \zeta = 0 \tag{6}$$

is required in place of conditions (5), then the two-point problem (2)-(4), and (6) will be a generalization of the problems /13/ $(m = 3)$, /5/ $(m = 2)$ on flows in wakes.

If the requirement

$$\Phi = b_1(-\zeta)^m + \ldots, \quad \zeta \to -\infty, \quad b_1 < 0 \tag{7}$$

is imposed in place of conditions (5), then (2)-(4) and (7) will be a generalization of the problems /2/ for $m = 1$ and /10/ for $m = 2$. They describe flows in mixing layers that separate two parallel streams moving to one side at different velocities.

The order of Eq.(2) is reduced if we set

$$f = d\Phi/d\zeta \; (d^2\Phi/d\zeta^2 = f df/d\xi, \; d^3\Phi/d\zeta^3 = f\,[(df/d\xi)^2 + f d^2f/d\xi^2], \; \xi \equiv \Phi)$$

To determine $f(\xi)$ a second-order differential equation is obtained with the boundary conditions

$$ff'' + f'^2 + \xi f' - \frac{m-1}{m} f = 0 \tag{8}$$

$$f = -c(\xi - c) - \frac{1}{4m}(\xi - c)^2 + O[(\xi - c)^3], \quad \xi \to c < 0 \tag{9}$$

$$\begin{gathered} f = m b^{1/m} \xi^{(m-1)/m} + \frac{m(m-1)(m-2)}{m+1} b^{2/m} \xi^{-2/m} + \\ O(\xi^{-1-3/m}) + D_1 \xi^{\varkappa_1} \exp\left[-\frac{b^{-1/m}}{m+1}\xi^{(m+1)/m}\right] + \ldots, \\ \xi \to +\infty, \quad \varkappa_1 = -\frac{2m^2 + 4m - 4}{m(m+1)}, \quad D_1 = \text{const} \end{gathered} \tag{10}$$

Condition (10) corresponds to (3), and (9) corresponds to (5). It is required to find a doubly continuous differentiable solution $f(\xi), \xi \in (c, \infty)$ of (8) that satisfies conditions (9) and (10). It follows from the group properties of problem (8)-(10) that if its solution exists and is unique, then the quantities c and b are connected by the relationship $k = m(-c)^{-(m+1)/m} b^{1/m}$ where k is some constant dependent on m.

The order of Eq.(8) is reduced, in turn, if the following substitution is made

$$f = \xi^2 F(\xi), \; \xi dF/d\xi = \Psi \tag{11}$$

We hence obtain

$$F\Psi \frac{d\Psi}{dF} = -\left(\Psi^2 + 7F\Psi + 6F^2 + \Psi + \frac{m+1}{m}F\right) \tag{12}$$

$$\begin{gathered} \frac{df}{d\xi} = \xi(\Psi + 2F), \quad \frac{d^2f}{d\xi^2} = -\left[(\Psi + 2F)^2 + \Psi + \frac{m+1}{m}F\right]\Big/ F \\ \frac{d^2\Phi}{d\zeta^2} = \xi^3 F(\Psi + 2F), \quad \frac{d^3\Phi}{d\zeta^3} = -\xi^4 F\left(\Psi + \frac{m+1}{m}F\right) \end{gathered} \tag{13}$$

The right-hand side of (12) and the numerator in the second formula of (13) are second-order polynomials. We denote them, respectively, by $P(F, \Psi)$ and $R(F, \Psi)$.

In order to construct a pattern of the behaviour of the triply continuous differentiable integral curves $\Phi(\zeta)$, as well as their corresponding integral curves $f(\xi)$, it is necessary to study the nature of the singular points of (12) and to find what requirements the integral curves $\Psi(F)$ must satisfy in order to satisfy the boundary conditions posed. Eq.(12) has three singular points

$$A(0, 0),\quad B(0, -1),\quad C\left(-\frac{m+1}{6m}, 0\right)$$

in a finite part of the plane (F, Ψ) and three infinitely remote points.

We connect each point of the plane (F, Ψ) to the centre of the unit hemisphere (located symmetrically about the plane and touching it at the origin) and we therefore set it in correspondence with a point on the hemisphere. Consequently, each infinitely remote singular point of (12) is stratified into two identical points on the equator that are symmetric relative to the centre. Those in which the curves enter or depart for $\Psi < 0$, we denote by Q, E, G, and for $\Psi > 0$ by Q_*, E_*, G_*.

The hemisphere is then projected on to a circle (see Figs.1–3). The axis $F = 0$ and the integral curves can only pass through the singularities.

The curve $P(F, \Psi) = 0$ for $m \neq 1/_5$ is a hyperbola at whose points $d\Psi/dF = 0$. For $m > 1/_5$ its branch P_1 passes through the points C and B, and P_2 through A. At points of the curve $R(F, \Psi) = 0$ which is a parabola and dissociates into the branches R_1 and R_2, $d^3f/d\xi^3$ vanishes (for $m = 1$ the parabola degenerates into two parallel lines $\Psi = -2F$, $\Psi = -2F - 1$). The branch R_1 always passes through the point B. The points on the curve P_i (or R_i will be denoted by (F, Ψ_{P_i}). As $F \to 0$ and $|F| \to \infty$ for $\Psi_{P_i}(F)$ we will have

$$d\Psi_{P_{1,2}}/dF = -1, -6, \quad |F| \to \infty$$

$$\Psi_{P_1} = -1 - \frac{6m-1}{m}F - \frac{5m-1}{m^2}F^2 + \ldots, \quad F \to 0$$

$$\Psi_{P_2} = -\frac{m+1}{m}F + \frac{5m-1}{m^2}F^2 + \ldots, \quad F \to 0$$

We denote the domains in which $d\Psi/dF < 0$ by $\Omega = \{F > 0,\ P(F, \Psi) < 0\}$ and $\Omega_* = \{F < 0,\ \Psi > 0,\ P(F, \Psi) < 0\}$.

We will now study each singularity separately.

The integral curves at the point A in a certain neighbourhood belong either to Ω or Ω_*. If $(F, \Psi) \in \Omega$ then a non-denumerable set of integrable curves enters the point A along the critical direction $\Psi = -(m+1)/mF$. Let $\mu(F)$ be one of them. Then as $F \to 0$ the following asymptotic relations /15/, /16/ will hold:

$$\Psi = \mu(F) + D_0 F^{\varkappa_0} \exp\left[-\frac{m}{m+1}F^{-1}\right] + \ldots \tag{14}$$

$$\mu(F) = -\frac{m+1}{m}F - \frac{(m-1)(m-2)}{m^2}F^2 + O(F^3)$$

$$\varkappa_0 = \frac{3m^2+4m-5}{(m+1)^2},\quad D_0 = \text{const}$$

If $m = 1, 2$, then $\Psi = -2F$, $\Psi = -3/_2 F$ are exact solutions of (12). It is hence most convenient to take them as $\mu(F)$.

If $F \to 0$ the following inequalities hold for the solution (14)

$$\Psi_{R_2}(F) < \Psi(F) < \Psi_{P_2}(F) < 0,\quad m > 1$$

$$\Psi(F) < \Psi_{R_2}(F) < \Psi_{P_2}(F) < 0,\quad 0 < m < 1$$

If $(F, \Psi) \in \Omega_*$, then a single integral curve that we call exceptional and denote by Ψ_A^* enters A.

The asymptotic behaviour of the solutions (14) for $\xi > 0$ corresponds to (10) in the plane (ξ, f). As $\zeta \to +\infty$ we will have in the plane (ζ, Φ)

$$\Phi = b(\zeta + l_A)^m - \frac{(m-1)(m-2)}{m+1}(\zeta + l_A)^{-1} + O[(\zeta + l_A)^{-(m+2)}] + \tag{15}$$

$$D_2(\zeta + l_A)^{\varkappa_2}\exp\left[-\frac{b}{m+1}(\zeta + l_A)^{m+1}\right] + \ldots$$

$$\varkappa_2 = -\frac{3m^2+5m-4}{m+1};\quad D_2,\ l_A \text{ is a const}$$

The point B is a saddle point. A single holomorphic curve representable by the following expansion as $F \to 0$ /17/ passes through it for any $m > 0$

$$\Psi = -1 + \sum_{k=1}^{\infty} b_k F^k;\quad b_1 = -\frac{6m-1}{2m} \tag{16}$$

$$b_2 = \frac{1}{3}[2b_1^2 + 7b_1 + 6], \quad b_k = \frac{1}{k+1}\Big[\frac{k+2}{2}\sum_{n=2}^{k} b_{n-1}b_{k-n+1} +$$
$$7b_{k-1}\Big], \quad k \geqslant 3; \quad \Psi \equiv \Psi_1 (F \geqslant 0), \quad \Psi \equiv \Psi_1^* \quad (F \leqslant 0)$$

The integral curves (9) correspond to it in the (ξ, f) plane for $c \neq 0$, and in the (ζ, Φ) plane as $c\zeta \to +\infty$

$$\Phi = c - (\operatorname{sign} c) \exp[-c(\zeta + l_B)] + \ldots, f > 0 \tag{17}$$
$$\Phi = c + (\operatorname{sign} c) \exp[-c(\zeta + l_B)] + \ldots, f < 0; \; l_B = \text{const}$$

For $m > m_*$ the singular point C is a focus. To a first approximation the integral curves in its neighbourhood behave thus:

$$\sqrt{u^2 + w^2} = h \exp\Big[\frac{m+7}{\sqrt{\varkappa}}\varphi\Big], \quad h = \text{const}$$
$$u = 2(m+1)\Psi + (m+7)\rho; \quad w = \varkappa\rho, \quad \varkappa = 23m^2 + 34m - 25$$
$$\rho = F + \frac{m+1}{6m}, \quad m_* = \frac{-17 + 12\sqrt{6}}{23} > \frac{1}{2}$$

Here φ is the polar angle in the (u, w) coordinate system. In the (ξ, f) plane we will have as $|\xi| \to \infty$

$$f = -\frac{m+1}{6m}\xi^2 + [C_1 \sin(\gamma \ln|\xi|) + C_2 \cos(\gamma \ln|\xi|)]|\xi|^{\omega} + \ldots$$
$$\omega = \frac{3(m-1)}{2(m+1)}, \quad \gamma = \frac{\sqrt{\varkappa}}{2(m+1)}$$

and in the (ζ, Φ) plane as $\zeta \to \zeta_c$

$$\Phi = \frac{6m}{m+1}(\zeta - \zeta_c)^{-1} + [C_3 \sin(\gamma \ln|\zeta - \zeta_c|) +$$
$$C_4 \cos(\gamma \ln|\zeta - \zeta_c|)] \cdot |\zeta - \zeta_c|^{1-\omega} + \ldots$$

Here and henceforth, ζ with a subscript is a certain finite value of ζ, while C with a subscript is a constant.

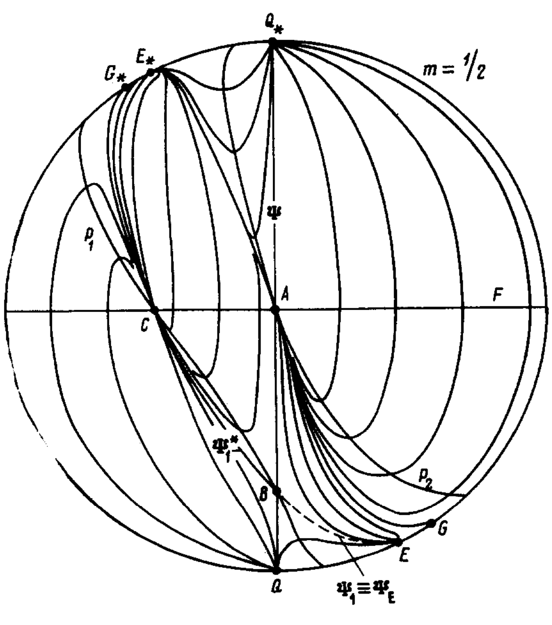

Fig.1

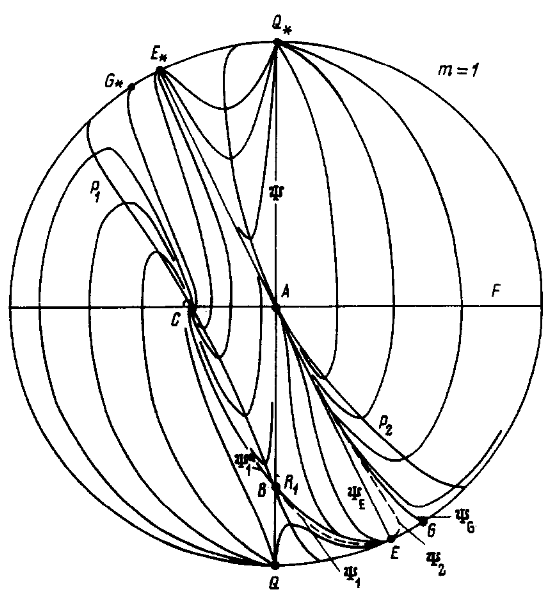

Fig.2

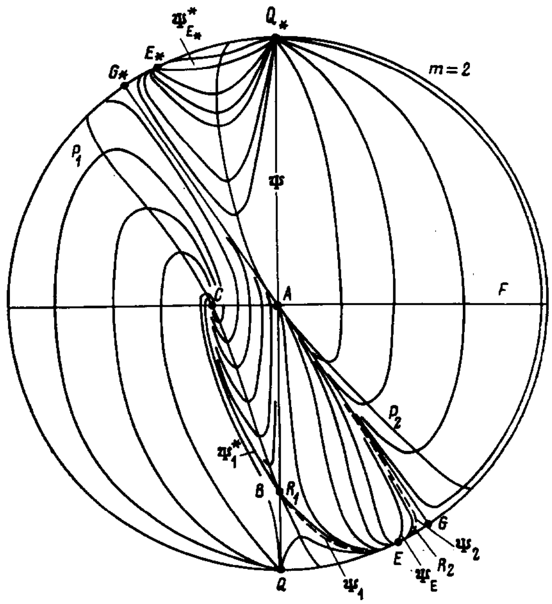

Fig.3

For $0 < m \leqslant m_*$ the point C becomes a node. The behaviour of the integral curves in its neighbourhood had the form

$$\Psi = -\frac{6}{\lambda_1}\left(F + \frac{m+1}{6m}\right) + h_1(\lambda_1 - \lambda_2)^{\lambda_1/\lambda_2}\left(F + \frac{m+1}{6m}\right)^{\lambda_1/\lambda_2} + \ldots, \quad h_1 = \text{const}$$

$$\Psi = -\frac{6}{\lambda_2}\left(F + \frac{m+1}{6m}\right) + \ldots, \quad \lambda_{1,2} = \frac{m+7 \pm \sqrt{-\varkappa}}{2(m+1)}$$

In the (ζ, Φ) plane this yields as $\zeta \to \zeta_c$

$$\Phi = \frac{6m}{m+1}(\zeta - \zeta_c)^{-1} + C_5|\zeta - \zeta_c|^{\lambda_2} + C_6|\zeta - \zeta_c|^{\lambda_1} + \ldots, \quad \lambda_1 \neq \lambda_2$$

$$\Phi = \frac{6m}{m+1}(\zeta - \zeta_c)^{-1} + C_7|\zeta - \zeta_c|^{\lambda_1}\ln|\zeta - \zeta_c| + C_8|\zeta - \zeta_c|^{\lambda_1} + \ldots, \quad \lambda_1 = \lambda_2$$

Therefore, incidence of an integral curve at the point C means that the solution will behave as $\Phi(\zeta) = O[(\zeta - \zeta_c)^{-1}]$ in the neighbourhood of a certain ζ_c.

To investigate the infinitely remote point E (and E_*), we make the following change of variables in (12)

$$2F - \frac{m-1}{m} = \frac{1}{t}, \quad \Psi = \frac{\sigma - 1}{t} \tag{18}$$

Consequently we obtain the equation

$$t(\sigma - 1)\left(\frac{m-1}{m}t + 1\right)\frac{d\sigma}{dt} = -\frac{\sigma}{2} + 2\sigma^2 + \frac{5m-3}{2m}\sigma t + \frac{(2m-1)(m-1)}{m^2}t^2 + \frac{m-1}{m}t\sigma^2$$

For $0 \leqslant t < r(C_E)$ its solution can be represented in the form of a convergent series ($r(C_E)$ is the radius of convergence) /17/

$$\sigma = \sum_{k=1}^{\infty} d_k t^{k/2}; \quad d_1 = C_E \text{ is arbitrary} \tag{19}$$

$$d_2 = -3C_E^2, \quad d_3 = -\frac{5}{4}e_3 - \frac{3m-2}{m}d_1$$

$$d_k = \frac{2}{k-1}\left[\frac{k-8}{4}e_k + \frac{m-1}{4m}(k-6)e_{k-2} - \frac{m(k+3)-(k+1)}{2m}d_{k-2} - \frac{(2m-1)(m-1)}{m^2}\delta_{k,4}\right], \quad k \geqslant 4; \quad e_k = \sum_{i=2}^{k} d_{i-1}d_{k-i+1}$$

The solution as $t \to 0, t \leqslant 0$ is constructed analogously. Therefore, the point E is a node. If the solution (19) is separated into two parts

$$\sigma = C_E t^{1/2} J_1(C_E, t) + J_2(C_E, t)$$
$$J_1 = \sum_{k=0}^{\infty} \frac{d_{2k+1}}{d_1} t^k, \quad J_2 = \sum_{k=1}^{\infty} d_{2k} t^k, \quad 0 \leqslant t < r$$

it can be shown that

$$J_1(C_E, t) = J_1(-C_E, t), \quad J_2(C_E, t) = J_2(-C_E, t)$$
$$d_{2k}(C_E) = d_{2k}(-C_E), \quad d_{2k+1}(C_E) = -d_{2k+1}(-C_E)$$

It is now necessary to find the relation between F and ξ. We introduce a new function τ by setting $t = \tau^2$. We obtain the equation

$$\xi \frac{d\tau}{d\xi} = \tau - \tau\sigma(\tau), \quad \sigma(\tau) = C_E \tau J_1(C_E, \tau^2) + J_2(C_E, \tau^2) \tag{20}$$

from (11) to determine $\tau = \tau(\xi)$.

According to the Briot-Bouquet theorem /17/, it admits of a denumerable set of holomorphic solutions possessing the property $\tau(\xi) \to 0$ as $\xi \to 0$

$$\tau = \xi \sum_{k=0}^{\infty} a_k \xi^k = \xi \varkappa(a_0, C_E, \xi), \quad 0 \leqslant |\xi| < R(a_0, C_E)$$

where $a_0 \neq 0$ is arbitrary and $R(a_0, C_E)$ is the radius of convergence of the series.

It follows from (20) that if $\tau_1(\xi, C_E)$ is a solution, then there exists a solution $\tau_2(\xi, C_E)$ such that $\tau_2(\xi, C_E) = -\tau_1(\xi, -C_E)$. For fixed $C_E = \alpha > 0$ and $\xi > 0$ we take $\tau_2 = \xi\varkappa(a_0, \alpha, \xi)$, $a_0 > 0$ for the integral curve (19). If $C_E = \beta < 0$ and $\xi < 0$, then we set the integral curve (19) in correspondence with the solution $\tau_1 = -\xi\varkappa(\bar{a}_0, -\beta, \xi)$. Now if we set $a_0 = \bar{a}_0$, $\beta = -\alpha$, then $\tau_1 = -\tau_2$, $0 \leqslant |\xi| < R$. We hence obtain by using (11), (18), that

$$f_1 = \frac{1}{2}\left[\frac{m-1}{m}\xi^2 + \varkappa^{-2}(a_0, \alpha, \xi)\right], \quad f_2 = \frac{1}{2}\left[\frac{m-1}{m}\xi^2 + \varkappa^{-2}(a_0, \alpha, \xi)\right]; \quad f_1 \equiv f_2, \quad 0 \leqslant |\xi| < R$$

Therefore, the passage from the integral curve (19) with the value $C_E = \beta < 0$ to the integral curve (19) with the value $C_E = \alpha = -\beta > 0$ through the point E in the (ξ, f) plane means that under the condition of continuity the appropriate integral curves $f(\xi)$ of (8) analytically continue the axis $\xi = 0$ at the point $(\xi = 0, f = f(0) \neq 0)$ for $f(0) \neq 0$. Discontinuous solutions of (8) at $\xi = 0$ are not considered. In the (ζ, Φ) plane the solution will behave as follows as $\zeta \to \zeta_e$:

$$\Phi = \frac{a_0^{-2}}{2}(\zeta - \zeta_e) + \frac{a_0^{-3}}{4}\alpha(\zeta - \zeta_e)^2 + \frac{m-1}{24m}a_0^{-4}(\zeta - \zeta_e)^3 + \ldots \tag{21}$$

Conversely, each triply continuously differentiable soltuion $\Phi(\zeta)$ in the neighbourhood of the point $\zeta = \zeta_e$, $\Phi = 0$ with $d\Phi/d\zeta_e > 0$ is mapped into the corresponding neighbourhood of the point E by two branches of the integral curve (19) with $C_E = \alpha > 0$ and $C_E = \beta = -\alpha < 0$.

If $\beta \neq -\alpha$ and the first derivative $\Phi'(\zeta)$ is continuous, then the second derivative $\Phi''(\zeta)$ will undergo a discontinuity on passing the point E. Now if the integral curve (19) is set in correspondence with the function τ_1 for $C_E > 0$ and $\xi < 0$, then the second derivative $\Phi''(\zeta)$ will be negative for $\zeta = \zeta_e$.

The solution of (2) in the neighbourhood of ζ_{e_*} will behave as follows on passing the point E_*:

$$\Phi = -\frac{a_0^{-2}}{2}(\zeta - \zeta_{e_*}) + \frac{a_0^{-3}}{4}C_{E_*}(\zeta - \zeta_{e_*})^2 + \frac{m-1}{24m}a_0^{-4}(\zeta - \zeta_{e_*})^3 + \ldots \tag{22}$$

The sign of the second derivative $\Phi''(\zeta)$ depends on the sign of the arbitrary constants $a_0 \neq 0$ and C_{E_*}.

We denote the integral curve issuing from E for $F > 0$ and described by (19) with $C_E = 0$ by Ψ_E. Its behaviour in the neighbourhood of ζ_e is described by the expansion (21) with $\alpha = 0$. The curve $\Psi_{E_*}^*$ is introduced analogously. Corresponding to it in the neighbourhood $\zeta = \zeta_{e_*}$ is (22) with $C_{E_*} = 0$.

The singular point G (as wll as G_*) is a saddle point. For $F > 0$ a single holomorphic curve

$$\Psi_G = -\frac{3}{2}F + \frac{m-2}{5m} + O(F^{-1}), \quad F \to +\infty, \quad m > 0$$

issues from it.

We will have, respectively, in the (ξ, f) and (ζ, Φ) planes

$$f = C_g |\xi|^{1/2} + \frac{2(m-2)}{15m}\xi^2 + \ldots, \quad \xi \to 0, \quad C_g > 0 \tag{23}$$
$$\Phi = C_g^2 (\zeta - \zeta_g)^2 + \frac{m-2}{30m} C_g^4 (\zeta - \zeta_g)^5 + \ldots, \quad \zeta \to \zeta_g + 0$$
$$\Phi = -C_g^2 (\zeta - \zeta_g)^2 + \frac{m-2}{30m} C_g^4 (\zeta - \zeta_g)^5 + \ldots, \quad \zeta \to \zeta_g - 0$$

The singular point Q (as well as Q_*) is a node. The integral curves in its neighbourhood are representable in the form

$$\Psi = \frac{C_q}{F} - 1 + \ldots, \quad F \to 0 \tag{24}$$

As $\xi \to c \neq 0$ in the plane (ξ, f), the function $f(\xi)$ will behave as follows:

$$f = \pm c^2 \left[\frac{2C_q}{c}(\xi - c)\right]^{1/2} - \frac{2c}{3}(\xi - c) + \ldots$$

If $\xi > c$, then $C_q c > 0$. Otherwise $C_q c < 0$. In the (ζ, Φ) plane this means that the point Q corresponds to a local extremum point of the analytic solution

$$\Phi = c + \frac{C_q c^3}{2}(\zeta - \zeta_q)^2 - \frac{C_q c^4}{6}(\zeta - \zeta_q)^3 + \ldots \tag{25}$$

It follows from the relationships found that the constant C_q in (24) should be kept as the axis $F = 0$ is crossed. For instance, let the integral curve (24) be incident at the point Q with $C_q < 0$ as $F \to +0$. To obtain the analytic solution (25) at the point $\zeta = \zeta_q$ it is necessary to go from Q_* along the curve (24) with $C_{q_*} = C_q$ as $F \to -0$. Only such continuations of the integral curves incident at Q or at Q_* will be examined below, and the "internal" transition from $Q (Q_*)$ to $Q_* (Q)$ will itself be denoted as $Q (Q_*) \to Q_* (Q)$.

Therefore, if the integral curve $\Psi(F)$ is incident for $F \geqslant 0$ at the singularity A for $\xi > 0$, then conditions (3) and (10) are satisfied, and (5) and (9) will be satisfied if the integral curve is incident in B for $\xi < 0$. Passage of the points E or E_* by the integral curve $\Psi(F)$ means that its corresponding integral curve $\Phi(\xi)$ intersects the ζ axis. If the points E and E_* are reached along Ψ_E and $\Psi^*_{E_*}$ then $\Phi(\zeta)$ and $d^2\Phi/d\zeta^2$ vanish simultaneously.

Let us investigate the boundary value problem (2)-(5). The integral curve Ψ_1^* issuing from the point B with $f < 0$, will always be incident at a point C for $0 < m < \infty$ (Figs.1-3). Consequently, we consider the behaviour of the curve Ψ_1 described in a certain neighbourhood of the point B by the expansion (16). As $F \to 0$ the integral curves Ψ_1 will behave in the (ζ, Φ) and (ξ, f) planes according to (17) and (9) for $c \neq 0$, and assure satisfaction of condition (5). We will first assume that $m > 1$. The following inequalities hold at the point B

$$\frac{d\Psi_{P_1}}{dF} < \frac{d\Psi_1}{dF} < \frac{d\Psi_{R_1}}{dF} \quad \left(\frac{1}{2} < m < \infty\right)$$

It therefore follows that Ψ_1 issues from B located above P_1 and below R_1. From the Cauchy existence and uniqueness theorem it follows /17/ that $\Psi_1(F) > \Psi_{P_1}(F)$ when $F \in (0, +\infty)$. Both branches of the parabola R_1 and R_2 are incident at E as $F \to \infty$. The inequalities

$$\frac{d\Psi}{dF} < \frac{d\Psi_{R_1}}{dF} < 0, \quad \frac{d\Psi}{dF} < \frac{d\Psi_{R_2}}{dF} < 0$$

hold at the points $(F, \Psi_R) \in \Omega$.

Their satisfaction means that $\Psi_1 < \Psi_{R_1}$ for all $F > 0$. The curve Ψ_1 does not coincide with Ψ_E since the latter is between R_1 and R_2 in the neighbourhood of E. By virtue of the Cauchy theorem Ψ_E cannot emerge from the domain Ω. As $F \to 0$ it is incident at the point A. Consequently, all the integral curves issuing from A below Ψ_E are incident at the point E. The curve Ψ_G in the neighbourhood of the point G is located below P_2. It follows from the Cauchy theorem that $\Psi_{P_2} > \Psi_G$ for all $F \in (0, \infty)$ and Ψ_G will enter A as $F \to 0$.

In the neighbourhood of the point E the curve Ψ_1 is described by the expansion (19) with a certain fixed $C_E = \beta_* < 0$. All the integral curves issuing from A and located between Ψ_G and Ψ_E are incident at the point E. Among them we extract the curve Ψ_2 with $C_E = \alpha_* = -\beta_*$. We let K denote the curve comprised of the branches Ψ_1 and Ψ_2. In the (ξ, f) plane integral curves in the class of twice continuously differentiable solutions of the problem (8)-(10) and issuing from points $(c, 0)$, $c < 0$ with asymptotic form (9) passing through the axis $\xi = 0$ and having the asymptotic form (10) as $\xi \to +\infty$, correspond to it. If the constant $b > 0$ is given, then a single curve is extracted from this set of integral curves $f(\xi)$, to which the solutions

$$\int_0^{\Phi} \frac{d\xi}{f(\xi)} = \zeta + L, \quad f > 0, \quad c < \Phi < \infty, \ c < 0 \tag{26}$$

correspond in the (ζ, Φ) plane.

The automatically satisfy conditions (3) and (5). As $\zeta \to -\infty$ the first formula of (17) with

$$l_B = -\int_0^c \left[\frac{1}{f(\xi)} + \frac{1}{c(\xi - c)}\right] d\xi - \frac{\ln|c|}{c} + L$$

will be valid, while for $\zeta \to +\infty$ we will have (15) with

$$l_A = -\int_0^{\infty} \left[\frac{1}{f(\xi)} - \frac{1}{mb^{1/m}\xi^{(m-1)/m}}\right] d\xi + L$$

The displacement thickness of the mixing layer is related to the value of the constant l_A. Only one out of all the solutions (26), with $L = 0$, will satisfy condition (4). Since the curve K lies completely outside the parabola R, and the line $\Psi = -2F$ is within it, it follows from relationships (11) and (13) that

$$\frac{df}{d\xi} > 0, \ \frac{d^2f}{d\xi^2} < 0, \ \xi \in (c, \infty); \ \frac{d\Phi}{d\zeta} > 0, \ \frac{d^2\Phi}{d\zeta^2} > 0$$
$$\zeta \in (-\infty, +\infty)$$

We will now assume that $1/2 < m \leqslant 1$ (Fig.2). As $F \to -\infty$ the branches of the parabola R are incident at the point E_*. When $m = 1$, it dissociates into two lines, $\Psi_{R_2} = -2F$ and $\Psi_{R_1} = -2F - 1$. It follows from the inequality $d\Psi/dF < -2$, that holds at points of the line $\Psi = -2F - 1$, that Ψ_1 is incident at E located below $\Psi = -2F - 1$ connecting B and E. The curve Ψ_2 from the point E is incident at the point A intersecting the line $\Psi = -2F$, which means a change in the sign of $d^2\Phi \mid d\zeta^2$ from positive to negative for a value $\zeta = \zeta_0$ such that $\Phi(\zeta_0) > 0$. The first derivative of the solution is $\Phi'(\zeta) > 0$. The behaviour of the third derivative is given in (13).

The value of the constant k is found from the formula

$$k = 2^{(m+1)/2m} \exp\left\{\frac{m+1}{m}\int_0^{\infty}\left[\frac{1}{\Psi_1} + \frac{1}{2(F+1)}\right] dF\right\} \chi(\Psi_2)$$

$$\chi(\Psi) = 2^{-(m+1)/2m} \exp\left\{-\frac{m+1}{m}\int_1^{\infty}\left[\frac{1}{\Psi} + \frac{1}{2F}\right] dF - \int_0^1 \left[\frac{m+1}{m\Psi} + \frac{1}{F}\right] dF\right\}$$

For $m = 1/2$ (Fig.1) the solution $\Psi = -2F - 1$ passes simultaneously through the points E, B, C, E_*. Its part Ψ_1 describes the escape of a plane jet from an orifice /18/. When $1/3 < m < 1/2$, the curve Ψ_1 is incident at E, then at Q. Its continuation emerges from Q_* and is incident at C. For $m = 1/3$ we have $\Psi_1 = \Psi_G = -3F/2 - 1$ and Ψ_1 is incident at G. For $0 < m < 1/3$ the curve Ψ_1 is incident at Q_*. Its continuation from the point Q is incident at C.

Therefore, for $m > 1/2$ and a given quantity $b > 0$ the solution of problem (2)-(5) in the class of triply continuously differentiable functions exists and is unique. There are no solutions for $0 < m \leqslant 1/2$.

Let us consider problem (2)-(4) and (6). For $m = 1/2$ the curves Ψ_E and $\Psi^*_{E_*}$ coincide with the line $\Psi = -2F - 1$ (Fig.1). If $0 < m < 1/2$, then they are incident at the point C. For $1/2 < m < 1$ the curve Ψ_E emerges from the point E below $\Psi = -2F - 1$. It follows from the inequality $d\Psi/dF < -2$, that holds at points of $\Psi = -2F - 1$, that $\Psi_E > -2F - 1$ for all $F \in (0, \infty)$. Since the inequality $d\Psi/dF > -2$ holds at points of the line $\Psi = -2F$, then $\Psi_E < -2F$ when $F \in (0, \infty)$. As $F \to 0$ the integral curve Ψ_E is incident at A. The final solution is given by (26) with $L = 0$. Hence

$$\Phi(\zeta) > 0, \ \Phi'(\zeta) > 0, \ \Phi''(\zeta) < 0, \ 0 < \zeta < \infty$$

In the neighbourhood of $\zeta = 0$ the solution is described by the expansion (21) ($\zeta_e = 0$, $\alpha = 0$) and satisfies conditions (4) and (6). As $\zeta \to \infty$ condition (3) is satisfied. For $0 < m < 1$ the curve $\Psi^{*\prime}_{E_*}$ is incident at the point C since it lies below, the exceptional curve $\Psi_A{}^*$. The latter issues from the point A, reaches the point E_* and lies above the line $\Psi = -2F$. This results from the inequality $d\Psi/dF < -2$ that holds at points of the line $\Psi = -2F$.

For $m = 1$ the curves Ψ_E and $\Psi^*_{E_*}$ coincide with the line $\Psi = -2F$ (Fig.2). In the physical plane the solution becomes trivial $\Phi = (a_0^{-2}\zeta)/2$. For $m > 1$ (Fig.3) the curve Ψ_E is always incident at the point A and formula (26) yields the solution of the problem for $\xi > 0$. The connection between the constants a_0 and b is given by the formula $a_0^{-1} = (m/k_1)^{m/(m+1)}b^{1/(m+1)}$, where $k_1 = \chi(\Psi_E)$ is a fixed constant obtained from the solution of (2)-(4), and (6) for $a_0 = 1$ in (21). The curve $\Psi^*_{E_*}$ lies above $\Psi_A{}^*$ and is incident at Q_* as $F \to -0$. Its continuation Ψ_{QEA} issues from Q, passes E and is incident in A. We denote the union of the integral curves $\Psi^*_{E_*}$ and Ψ_{QEA} by Ψ_{E_*} and, for convenience, also call it a curve since the points Q_* and Q are identical.

The curves $\Phi(\zeta)$ corresponding to Ψ_{E_*} and satisfying the conditions of the boundary value problem (2)-(4) and (6) behave as follows. As $\zeta \to +0$ they are described by the expansion (22) with $\zeta_{e_*} = 0$, $C_{E_*} = 0$. When ζ reaches the value $\zeta_q > 0$ the derivative $d\Phi/d\zeta$ changes sign from negative to positive. In the neighbourhood of ζ_q the solution is described by the expansion (25) with $C_q < 0$, $c < 0$. Then it vanishes at the point ζ_e and satisfies (15) as $\zeta \to +\infty$. From the group properties of (2), (8), the relations

$$a_0^{-1} = \left(\frac{m}{k_2}\right)^{1-\varkappa_3} b^{\varkappa_3}, \quad \zeta_{q,e} = a_0 \bar{\zeta}_{q,e}, \quad \varkappa_3 = (m+1)^{-1}$$

follow.

Here k_2, $\bar{\zeta}_q$, $\bar{\zeta}_e$ are fixed constants obtained from the solution of problem (2)-(4) and (6) when $a_0 = 1$ in (22). If $f_1(\xi)$ denotes the solution of (8) corresponding to Ψ_{E_*} for $F \leqslant 0$, and $f_2(\xi)$ denotes the solution for $F \geqslant 0$, then the integral curve in the (ζ, Φ) plane will be described by the formulas

$$\zeta = \int_0^{\Phi} \frac{d\xi}{f_1(\xi)}, \quad c \leqslant \Phi \leqslant 0, \quad 0 \leqslant \zeta \leqslant \zeta_q, \quad \zeta(c) = \zeta_q \tag{27}$$

$$\zeta = \zeta_q + \int_c^{\Phi} \frac{d\xi}{f_2(\xi)}, \quad c \leqslant \Phi < \infty, \quad \zeta_q \leqslant \zeta < \infty$$

It is seen from (27) that ζ is a monotonic function if we move along the curve Ψ_{E_*} from E_* to A and one value of Φ corresponds to each value of ζ. The second derivative $\Phi''(\zeta)$ is positive everywhere. The function $\Phi(\zeta)$ inverse to (27) yields the solution of problem (2)-(4) and (6).

Therefore, integral curve $\Phi(\zeta)$ exist that correspond to both Ψ_E and Ψ_{E_*} and yield a solution of the boundary value problem (2)-(4) and (6). Its non-uniqueness was first indicated for $m = 3$ in /13/. The problem is investigated numerically in /14/.

The uniqueness can be ensured if the additional requirement $\Phi'(0) > 0$ /1/ is imposed on the solution since from (22) we have $\Phi'(0) < 0$ for the curve Ψ_{E_*}

It follows from the above that for a given $b > 0$ the solution of problem (2)-(4) and (6) in the class of triply continuously differentiable functions exists and is unique for $1/2 < m \leqslant 1$. For $m > 1$ two solutions exist. One is characterized by a positive first derivative $d\Phi/d\zeta > 0$, $0 \leqslant \zeta < \infty$. The other is the presence of a domain $[0, \zeta_q)$, $\zeta_q > 0$ in which $d\Phi/d\zeta < 0$. Consequently, a unique solution can be extracted by using an additional demand, namely, giving the sign of the derivative at zero /1/. There are no solutions for $0 < m \leqslant 1/2$.

We will now examine problem (2)-(4) and (7) briefly. For $0 < m \leqslant 1/2$ it has no solution. For $1/2 < m \leqslant 1$ the curves issuing from A below Ψ_2 are incident at E and then return back to A (Fig.2). They yield the solution of (2)-(4) and (7) with $d\Phi/d\zeta > 0$. By using the selection of the constants C_E a solution of the Lock problem formulated in /3/ can also be constructed. For $1 < m \leqslant 2$ solutions besides those mentioned above still exist that satisfy conditions (3), (4) and (7). This is part of the curves (but all for $m = 2$ that emerge from A below Ψ_G but above Ψ_2 (Fig.3). They pass E, are then incident at $Q \to Q_*$, E_* and return back to A through $Q_* \to Q$, E. For $m > 2$ there are also those curves that, issuing from the point A, are incident at E then at $Q \to Q_*$ and return back to A. For $1 < m < 2$ the curve Ψ^*_A continued by Ψ_{QEA} also yields a soltuion of the problem if $b_1 > 0$ and its magnitude is in agreement with b. It describes a mixing layer that separates two parallel streams moving on different sides at different velocities. The solvability of (2)-(4) and (7) for arbitrary $b > 0$ and $b_1 < 0$ is not investigated here.

We merely note that the curve Ψ_G yields the Blasius boundary layer solution for $m = 1$ (Fig. 2). In the neighbourhood of $\zeta = 0$ it behaves as (23) with $\zeta_g = 0$, $\zeta \geqslant 0$, and as (15) as $\zeta \to \infty$.

In conclusion, we consider the solution of (2) when $m = \infty$ and

$$\Phi(\zeta) = e^{\zeta} + \ldots, \quad \zeta \to \infty \tag{28}$$

is required instead of (3).

Problem (2), (4), (5) and (28) arises in the theory of non-stationary separation /12/. We will show that its solution exists and is unique. The nature of the singular points of Eq.(12) does not alter for $m = \infty$. For $C_E = -1/\sqrt{2}, \xi < 0$ and $C_E = 1/\sqrt{2}, \xi > 0$ we obtain

$$\sigma_{1,2} = -\frac{1}{2}[3t \pm \sqrt{t(3t+2)}], \quad \Psi_{1,2} = -\frac{1}{2}[1 + 4F \pm \sqrt{1+4F}]$$

$$\tau_{1,2} = \mp \frac{d_0\xi}{\sqrt{2 + 2d\xi - d^2\xi^2}}, \quad t_{1,2} = \tau_{1,2}^2, f_{1,2} = \frac{1}{d^2} + \frac{\xi}{d}, \quad d > 0$$

It hence follows that $f_1 \equiv f_2$ for all ξ. We find from the boundary condition for $f(\xi)$ as $\xi \to +\infty$ that $d = 1$. From (26) with $L = 0$ we obtain the solution $\Phi = e^{\zeta} - 1$ /12/. It is unique since the curve Ψ_1^* issuing from B with $f < 0$ is at once incident at A. The solutions

$$\Phi = -d^{-1}\left(\exp\frac{\zeta - \zeta_0}{d} + 1\right), \quad d \neq 0$$

which do not satisfy the boundary conditions, correspond to it.

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